

**APPROACHES TO THE MONOPOLE–DYNAMIC DIPOLE
VACUUM SOLUTION CONCERNING THE STRUCTURE OF
ITS ERNST’S POTENTIAL ON THE SYMMETRY AXIS**

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Abstract The FHP algorithm [1] allows to obtain the relativistic multipole moments of a vacuum stationary axisymmetric solution in terms of coefficients which appear in the expansion of its Ernst’s potential ξ on the symmetry axis. First of all, we will use this result in order to determine, at a certain approximation degree, the Ernst’s potential on the symmetry axis of the metric whose only multipole moments are mass and angular momentum.

By using Sibgatullin’s method [2] we then analyse a series of exact solutions with the afore mentioned multipole characteristic; besides, we present an approximate solution whose Ernst’s potential is introduced as a power series of a dimensionless parameter. The calculation of its multipole moments allows us to understand the existing differences between both approximations to the proposed pure multipole solution.

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1. INTRODUCTION

As it is well known, the relativistic multipole moments defined by Geroch [3] and Hansen [4] allow us to characterize, in a unique manner, vacuum stationary metrics. Particularly, Schwarzschild's solution (spherically symmetric) can be described as that one whose unique multipole moment is monopole.

In [5] we have introduced a static and axisymmetric solution of the Einstein vacuum field equations whose only multipole moments are mass and the quadrupole moment. Our aim in the present work is to generalize the above mentioned result to the non-static case, by searching the stationary and axisymmetric vacuum solution which in addition to mass possesses only a dynamic dipole moment.

Fodor, Hoenselaers and Perjés [1] have developed an algorithm to calculate multipole moments in terms of the coefficients arising in the expansion of the Ernst's potential ξ on the symmetry axis as a power series on the inverse of the Weyl's coordinate z . According to this result it is possible to determine such coefficients corresponding to a solution with the mentioned multipole characteristics.

There is a method, due to Sibgatullin [2], which allows to generate exact solutions of vacuum, stationary and axisymmetric field equations from the Ernst's potential on the symmetry axis, whenever its structure is a polynomial ratio. This method has been broadly used [6], [7], [8] and it has been recently completed by the introduction of some general expressions which make its use much easier [9].

Our first aim, is to prove that a finite set of coefficients which appear in the expansion of the potential ξ on the symmetry axis is sufficient enough to describe a potential $E \equiv (1 - \xi)/(1 + \xi)$ of rational type on the symmetry axis. Therefore, the use of Sibgatullin's method, makes it possible to construct an Ernst's potential E in terms of these coefficients. Nevertheless, if we look for an exact solution of the $M - J$ type, then the result of this condition leads to the Ernst's potential on the symmetry axis by means of a ratio of series. Hence, there is no finite number of coefficients which describe the Ernst's potential of Monopole-Dynamic dipole solution as a polynomial ratio.

We then approach the $M - J$ solution as the limit of a sequence of exact solutions which possess a progressively smaller number of multipole moments higher than the dipole. In spite of this, the behaviour of the multipole moments shows that the mentioned series approaches the $M - J$ solution in a rather unexpected way.

In order to obtain an alternative approach to the $M - J$ solution, we will propose a series of approximate solutions described by the partial sums of the

expansions of the Ernst's potential ξ as a power series in a dimensionless parameter \mathcal{J} . To do that, we use Schwarzschild's solution as a seed solution and solve Ernst's equation at successive orders in the parameter and impose the corresponding structure of the $M - J$ solution on the symmetry axis.

As we will see, the order of magnitude of its multipole moments decreases as the order of approximation rises. That leads us to conclude that the parameter \mathcal{J} controls the successive solutions and contributes to give a physical meaning to the approximate solution.

A more realistic stationary solution would be a $M - Q - J$ solution, i.e., the one having only mass, massive quadrupole moment and dynamic dipole, since a rotating object flattens and, hence, all its massive multipole moments represent such deviation from sphericity. Nevertheless, we can imagine so rigid an object that the $M - J$ solution itself would be physically relevant. Besides, since the static case is already solved by $M - Q$ solution, we want to discuss $M - J$ solution and consider the $M - Q - J$ solution as a generalization of both solutions.

2. STRUCTURE OF THE M-J SOLUTION ON THE SYMMETRY AXIS

Let us be ξ the Ernst's potential of a stationary axisymmetric solution of vacuum Einstein's field equations [10]

$$(\xi\xi^* - 1)\Delta\xi = 2\xi^*(\nabla\xi)^2 \quad , \quad (1)$$

being $\xi \equiv \frac{1 - E}{1 + E}$, where the Ernst's potential E is the complex function whose real part represents the norm of the Killing vector describing stationarity. On symmetry axis, this potential ξ can be expanded by means of a power series of the inverse Weyl's coordinate z as follows:

$$\xi(\rho = 0, z) = \sum_{n=0}^{\infty} m_n z^{-(n+1)} \quad (2)$$

where ρ represents the Weyl's radial coordinate.

Fodor, Hoenselaers and Perjés [1] have developed an algorithm which allows to calculate the Geroch [3] and Hansen [4] relativistic multipole moments, related to a vacuum stationary axisymmetric solution, in terms of the coefficients m_n arising in the previous expansion (2). Both the result obtained up to multipole order 10 by the afore mentioned authors, and the calculations we have carried

out up to order 20 lead us to show that the relation between multipole moments and coefficients m_n is triangular. That is to say, the multipole moment and the corresponding coefficient m_n at every order, differ in a certain combination of lesser order m_k coefficients. Therefore, these relations enable us to determine unequivocally the coefficients m_h which allows to outline the expansion of ξ in terms of the known multipole moments for any given solution.

So we have obtained that the solution having only massive monopole and dynamic dipole is characterized by an Ernst's potential ξ whose expansion on the symmetry axis provides the following coefficients m_n up to the order 20,

$$\begin{aligned}
m_0 &= M \quad , \quad m_1 = iJ \\
m_2 &= 0 \quad , \quad m_3 = 0 \\
m_4 &= \frac{1}{7}MJ^2 \\
m_5 &= -\frac{1}{21}iJ^3 \\
m_6 &= \frac{1}{21}M^3J^2 \\
m_7 &= -\frac{13}{231}M^2iJ^3 \\
m_8 &= \frac{5}{231}M^5J^2 - \frac{40}{3003}MJ^4 \\
m_9 &= -\frac{115}{3003}M^4J^3i - \frac{4}{3003}J^5i \\
m_{10} &= \frac{5}{429}M^7J^2 - \frac{115}{7007}M^3J^4 \\
m_{11} &= -\frac{1}{39}M^6J^3i - \frac{389}{357357}M^2J^5i \\
m_{12} &= \frac{1}{143}M^9J^2 - \frac{1569}{119119}M^5J^4 - \frac{53}{2909907}MJ^6 \\
m_{13} &= -\frac{43}{2431}M^8J^3i - \frac{265}{108927}M^4J^5i - \frac{13051}{20369349}J^7i \\
m_{14} &= \frac{1}{221}M^{11}J^2 - \frac{187618}{20369349}M^7J^4 + \frac{1129}{10968111}M^3J^6 \\
m_{15} &= -\frac{53}{4199}M^{10}J^3i - \frac{10954}{2263261}M^6J^5i - \frac{831513}{364385021}M^2J^7i
\end{aligned}$$

$$\begin{aligned}
m_{16} &= \frac{1}{323} M^{13} J^2 - \frac{40346}{6789783} M^9 J^4 - \frac{454}{15954939} M^5 J^6 - \frac{2419504}{16397325945} M J^8 \\
m_{17} &= -\frac{3}{323} M^{12} J^3 i - \frac{12480070}{1717815099} M^8 J^5 i - \frac{293822614}{60123528465} M^4 J^7 i - \\
&\quad - \frac{35634548}{147575933505} J^9 i \\
m_{18} &= \frac{5}{2261} M^{15} J^2 - \frac{862123}{245402157} M^{11} J^4 - \frac{10703470}{36074117079} M^7 J^6 - \\
&\quad - \frac{11186023022}{21103358491215} M^3 J^8 \\
m_{19} &= -\frac{173594465986}{21103358491215} M^6 J^7 i - \frac{99041865574}{87428199463605} J^9 M^2 i - \frac{365}{52003} M^{14} J^3 i - \\
&\quad - \frac{6808829}{736206471} M^{10} J^5 i \\
m_{20} &= \frac{5}{3059} M^{17} J^2 - \frac{6540151}{3681032355} M^{13} J^4 - \frac{9908406983}{21103358491215} M^9 J^6 - \\
&\quad - \frac{4674899812546}{4283981773716645} M^5 J^8 - \frac{985078066594}{18971919283602285} M J^{10} \quad ,
\end{aligned} \tag{3}$$

where M and J represent Mass and Angular Momentum respectively.

These expressions suggest that the coefficients m_n of the potential ξ representing the Monopole-Dynamic dipole solution can be written in the following way

$$\begin{aligned}
m_{2k} &= M^{2k+1} \sum_{n=1}^{(2k+1)/4} \mathcal{J}^{2n} G(2n, 2k) \\
m_{2k+1} &= M^{2k+2} \sum_{n=1}^{k/2} \mathcal{J}^{2n+1} G(2n+1, 2k+1)
\end{aligned} \quad , \quad (k = 0, 1, \dots) \tag{4}$$

where we have introduced a dimensionless parameter $\mathcal{J} \equiv \frac{m_1}{m_0^2} = i \frac{J}{M^2}$ and the function $G(l, h)$. For every coefficient m_h this function describes the numerical factor multiplying the power l of the parameter \mathcal{J} .

By substituting the expressions (4) in the expansion (2) of the Ernst's potential ξ on the symmetry axis, and rearranging sums, it is possible to write this potential as a power series of the parameter \mathcal{J} :

$$\xi(\rho = 0, z) = \frac{M}{z} + \mathcal{J} \frac{M^2}{z^2} + \sum_{\alpha=2}^{\infty} \mathcal{J}^{\alpha} \Phi_{\alpha} \quad , \tag{5}$$

where functions Φ_α are defined below

$$\Phi_{2n} = \sum_{k=2n}^{\infty} G(2n, 2k) \hat{\lambda}^{2k+1} \quad (6a)$$

$$\Phi_{2n+1} = \sum_{k=2n}^{\infty} G(2n+1, 2k+1) \hat{\lambda}^{2k+2} \quad , \quad (6b)$$

with the notation $\hat{\lambda} \equiv \frac{M}{z}$. Let us note that since the parameter \mathcal{J} is imaginary, the functions Φ_α with an odd index turns the series (5) into an imaginary function.

As shown in (5), the Ernst's potential ξ can be expressed on the symmetry axis by a double series; one of them in terms of the parameter \mathcal{J} and the other being a power series of the inverse coordinate z . Nevertheless, as we will see, the sum of the series Φ_α can be obtained, at least up to first orders. In order to do that it is necessary to obtain the analytic expressions which describe the double index functions $G(\alpha, h)$. If the first of those indexes is fixed, that is to say, if we consider a certain value for the power of the parameter \mathcal{J} , we have tried to adjust the resulting series of the corresponding terms arising from every coefficient m_n .

For example, it is very easy to check that factors appearing with powers two and three in the parameter \mathcal{J} verify respectively the following expressions

$$G(2, 2k) = \frac{15}{(2k+3)(2k+1)(2k-1)} \quad (7)$$

$$G(3, 2k+1) = \frac{15(10k-17)}{(2k+5)(2k+3)(2k+1)(2k-1)}$$

Now, it is quite simple to obtain the sum of the series Φ_α by rewriting the functions $G(\alpha, h)$ as a sum of irreducible fractions. Particularly, for $G(2, h)$ we have

$$G(2, 2j+1) = 0$$

$$G(2, 2j) = \frac{15}{8} \left[\frac{1}{2j+3} - \frac{2}{2j+1} + \frac{1}{2j-1} \right] \equiv \sum_{i=0}^2 \frac{g_i^{(2)}}{2i+2j-1} \quad (8)$$

Taking this expression into (6a) the function Φ_2 gives

$$\Phi_2 = \sum_{k=2}^{\infty} \sum_{j=0}^2 \frac{g_j^{(2)}}{2k+2j+1} \hat{\lambda}^{2k+1} \quad . \quad (9)$$

Rearranging sums and making use of Lemma 3 of Appendix, we obtain the following finite sum:

$$\Phi_2 = \sum_{j=0}^2 g_j^{(2)} \hat{\lambda}^4 \sum_{k=0}^{3-j} C_{2(3-j), 2k} Q_{2k}(1/\hat{\lambda}) \quad , \quad (10)$$

where the coefficients C_{lh} are defined in Appendix and the functions $Q_h(x)$ are special Legendre's functions of second kind. Let us note that the previous expression can be written as follows

$$\Phi_2 = \hat{\lambda}^4 \sum_{k=0}^3 Q_{2k}(1/\hat{\lambda}) \sum_{j=0}^{\min(3-k,2)} g_j^{(2)} C_{2(3-j),2k} \quad . \quad (11)$$

3. SEQUENCE OF EXACT SOLUTIONS

The expression of Ernst's potential on the symmetry axis can be used as boundary condition to obtain solutions of the Ernst's equation. For example, Sibgatullin's method [2] simplifies this problem by solving a linear system of integral equations. So, the Ernst's potential E results from the following expression

$$E = \frac{1}{\pi} \int_{-1}^1 \frac{\mu(\sigma)e(\tau)}{\sqrt{1-\sigma^2}} d\sigma \quad , \quad (12)$$

where τ is a complex variable defined from the cylindrical Weyl's coordinates $\tau \equiv z + i\rho\sigma$, and $\sigma \in [-1, 1]$ is an arbitrary integration variable. Function $e(z)$ represents the value of the Ernst's potential E on the symmetry axis, that means, $e(z) \equiv E(\rho = 0, z)$. At length, the function $\mu(\sigma)$ must be a solution verifying the following integral equations system

$$\wp \int_{-1}^1 \frac{h(\tau, \eta)\mu(\sigma)}{(\tau - \eta)\sqrt{1-\sigma^2}} d\sigma = 0 \quad (13a)$$

$$\frac{1}{\pi} \int_{-1}^1 \frac{\mu(\sigma)}{\sqrt{1-\sigma^2}} d\sigma = 1 \quad , \quad (13b)$$

being η a complex variable defined as $\eta \equiv z + i\rho\varsigma$, with $\varsigma \in [-1, 1]$, and where the symbol \wp stands for the taking of the principal part of the integral. On the other hand, the function $h(\tau, \eta)$ is defined as is shown below

$$h(\tau, \eta) \equiv e(\tau) + \tilde{e}(\eta) \quad , \quad (14)$$

where the function $\tilde{e}(\eta)$ is obtained from $e(\eta)$ by conjugating first the variable, $\eta \rightarrow \eta^*$, and then the function, i.e., $\tilde{e}(\eta) = e^*(\eta^*)$.

Obviously, the general solution of the equations in (13) is not evident. Nevertheless, rather compact expressions have been obtained [9] for the Ernst's potential when the boundary condition $e(z)$ is a rational function, i.e,

$$E(\rho = 0, z) \equiv e(z) = \frac{P(z)}{Q(z)} \quad , \quad (15)$$

where $P(z)$ and $Q(z)$ are polynomials in the variable z , which, taking into account that the Ernst's potential must tend to 1 in the neighbourhood of infinity, should be as follows

$$\begin{aligned} P(z) &= z^N + \sum_{k=1}^N a_k z^{N-k} \\ Q(z) &= z^N + \sum_{k=1}^N b_k z^{N-k} \end{aligned} \quad (16)$$

Since we know the structure of the potential ξ on the symmetry axis in terms of coefficients m_n , immediately a question arises: whether there exists a relation between these coefficients and those of the polynomials in (16). The above question has been answered in [11] and in what follows we give the resulting expressions for $P(z)$ and $Q(z)$ in terms of m_n :

$$P(z) = (L_N)^{-1} \begin{vmatrix} z^N - \sum_{n=0}^{N-1} m_n z^{N-1-n} & m_N & \dots & m_{2N-1} \\ z^{N-1} - \sum_{n=0}^{N-2} m_n z^{N-2-n} & m_{N-1} & \dots & m_{2N-2} \\ \dots & \dots & \dots & \dots \\ z - m_0 & m_1 & \dots & m_N \\ 1 & m_0 & \dots & m_{N-1} \end{vmatrix}, \quad (17)$$

$$Q(z) = (L_N)^{-1} \begin{vmatrix} z^N + \sum_{n=0}^{N-1} m_n z^{N-1-n} & m_N & \dots & m_{2N-1} \\ z^{N-1} + \sum_{n=0}^{N-2} m_n z^{N-2-n} & m_{N-1} & \dots & m_{2N-2} \\ \dots & \dots & \dots & \dots \\ z + m_0 & m_1 & \dots & m_N \\ 1 & m_0 & \dots & m_{N-1} \end{vmatrix}, \quad (18)$$

where the following determinant L_N has been defined:

$$L_N \equiv \begin{vmatrix} m_{N-1} & m_N & \dots & m_{2N-2} \\ m_{N-2} & m_{N-1} & \dots & m_{2N-3} \\ \dots & \dots & \dots & \dots \\ m_0 & m_1 & \dots & m_{N-1} \end{vmatrix}. \quad (19)$$

Hence, from $2N$ coefficients m_k , it is possible to build on the symmetry axis the Ernst's potential E of a vacuum solution, which is a ratio of order N polynomials.

Let us consider now the following question: is potential E of the Monopole-Dynamic dipole solution a polynomial ratio on the symmetry axis?. On handling

the coefficients m_n in (3) corresponding to such solution, the determinants L_n (19) seem to be unlikely to be zero originating from some order N onwards. Since the coefficients m_n are not available for every order, we can only assert that the behaviour of the Ernst's potential of such solution does not correspond to a ratio of polynomials of order $N \leq 10$.

In spite of last statement, it is possible to construct a set of rational type potentials E on the symmetry axis involving these coefficients m_n . Thus, by Sibgatullin's method one can obtain a sequence of exact solutions which, as will be shown, approach the Monopole-Dynamic dipole solution.

The coefficients m_n (3) have been obtained on condition that the multipole moments higher than the dipole are zero. Hence, on fixing N coefficients m_n , we will get an Ernst potential which describes a solution whose $N - 2$ multipole moments higher than Angular momentum are zero. At the same time, multipole moments of higher order, although different from zero, are determined by just those N coefficients m_k .

In order to perform the sequence of exact solutions, let us proceed to consider the Ernst's potential on the symmetry axis as a ratio of polynomials whose order N will be progressively increased. Hence, at each stage, we will be fixing an increasingly bigger $2N$ number of multipole moments for the corresponding solution.

A) ORDER $N = 1$

Let us suppose first that Ernst's potential on the symmetry axis is a ratio of polynomials of order $N = 1$, i.e.,

$$e^{(1)}(z) = \frac{z + a_1}{z + b_1} \equiv \frac{P^{(1)}(z)}{Q^{(1)}(z)} . \quad (20)$$

In order to calculate coefficients a_1 and b_1 we handle the two first coefficients m_k in (3),

$$m_0 \equiv M \quad , \quad m_1 \equiv iJ . \quad (21)$$

We obtain polynomials $P^{(1)}(z)$ and $Q^{(1)}(z)$ in terms of these coefficients by using expressions (17) and (18) respectively, and as a result the Ernst's potential is written on the symmetry axis in the following way

$$e^{(1)}(z) = \frac{z - M - iJ/M}{z + M - iJ/M} . \quad (22)$$

The previous expression is exactly the corresponding potential of Kerr's metric with parameters M and $a \equiv J/M$. By using Sibgatullin's method and the expressions in [11], it is possible to obtain the Ernst's potential for every range of Weyl's

coordinates $\{\rho, z\}$, as follows

$$E^{Kerr} = \frac{\alpha(r_+ + r_-) + ia(r_+ - r_-) - 2\alpha M}{\alpha(r_+ + r_-) + ia(r_+ - r_-) + 2\alpha M} \quad , \quad (23)$$

being α the positive root of polynomial $P(z)\tilde{Q}(z) + \tilde{P}(z)Q(z)$ (numerator of the function $h(z, z)$ (14)), i.e.,

$$\alpha_{\pm} = \pm \sqrt{M^2 - a^2} \quad (24)$$

and where $r_{\pm} = \sqrt{\rho^2 + (z - \alpha_{\pm})^2}$

Now, let us calculate the coefficients m_k higher than those proposed in (21). To do that, we can use some expressions in [11] which relate such coefficients to those of the polynomials $P^{(1)}(z)$ and $Q^{(1)}(z)$, and gives

$$m_k = M(ia)^k \equiv M^{k+1} \mathcal{J}^k \quad . \quad (25)$$

It must be remembered that one property of Kerr's metric turns out to be the identity between its multipole moments and the coefficients m_k entering the expansion of the potential ξ on the symmetry axis [1], and leads to,

$$\begin{aligned} M_0 &= M \quad , & M_1 &= \mathcal{J}M^2 \quad , & M_2 &= \mathcal{J}^2M^3 \quad , & M_3 &= \mathcal{J}^3M^4 \quad , \\ M_4 &= \mathcal{J}^4M^5 \quad , & M_5 &= \mathcal{J}^5M^6 \quad , & M_6 &= \mathcal{J}^6M^7 \quad . \end{aligned} \quad (26)$$

Obviously, the coefficients m_n higher than m_2 do not equal the corresponding coefficients of the $M - J$ solution. That is a good reason to step forward.

B) ORDER $N = 2$

Let us consider the Ernst's potential on the symmetry axis as a ratio of polynomials of order $N = 2$, i.e.,

$$e^{(2)}(z) = \frac{z^2 + a_1z + a_2}{z^2 + b_1z + b_2} \equiv \frac{N^{(2)}(z)}{D^{(2)}(z)} \quad . \quad (27)$$

We introduce four coefficients m_k , according to expressions (3) with the following values

$$m_0 \equiv M \quad , \quad m_1 \equiv iJ \quad , \quad m_2 = m_3 = 0 \quad . \quad (28)$$

If we choose the coefficients m_k in (28), the solution we generate will have the quadrupole moment and the octupole moment equal to zero since m_2 and m_3 are just equal to these multipole moments respectively, .

The calculations of the polynomials $P^{(2)}(z)$ and $Q^{(2)}(z)$ leads to the following result

$$e^{(2)}(z) = \frac{z^2 - Mz - iJ}{z^2 + Mz + iJ} \quad . \quad (29)$$

In order to construct the Ernst's potential according to Sibgatullin's method it is necessary to obtain the roots of the function $h(z, z) \equiv P(z)\tilde{Q}(z) + \tilde{P}(z)Q(z)$. For this case, two roots of that function turns out to be real numbers whereas the other two are imaginary conjugated numbers:

$$\begin{aligned} \alpha_1^\pm &= \pm \frac{M}{\sqrt{2}} \sqrt{1 + \sqrt{1 - 4\mathcal{J}^2}} = \pm \frac{M}{2} \left[\sqrt{1 + 2\mathcal{J}} + \sqrt{1 - 2\mathcal{J}} \right] \\ \alpha_2^\pm &= \pm \frac{M}{\sqrt{2}} \sqrt{1 - \sqrt{1 - 4\mathcal{J}^2}} = \pm \frac{M}{2} \left[\sqrt{1 + 2\mathcal{J}} - \sqrt{1 - 2\mathcal{J}} \right] \end{aligned} \quad . \quad (30)$$

From these roots it is possible [11] to write out the Ernst's potential E as follows

$$E^{(2)}(\rho, z) \equiv \frac{\Lambda + \Gamma}{\Lambda - \Gamma} \quad , \quad (31)$$

$$\begin{aligned} \Lambda &\equiv \frac{1}{8} r_1^+ \left[\left(\frac{p-1}{p} \right)^2 r_2^+ + \left(\frac{\bar{p}+1}{\bar{p}} \right)^2 r_2^- \right] \\ &+ \frac{1}{8} r_1^- \left[\left(\frac{\bar{p}-1}{\bar{p}} \right)^2 r_2^+ + \left(\frac{p+1}{p} \right)^2 r_2^- \right] + \frac{1}{4} \frac{p^2 \bar{p}^2 - 1}{p^2 \bar{p}^2} (r_1^- r_1^+ + r_2^- r_2^+) \end{aligned} \quad (32)$$

$$\begin{aligned} \Gamma &\equiv \frac{M}{8} (p - \bar{p}) \left[\left(\frac{p-1}{p} \right) \left(\frac{\bar{p}+1}{\bar{p}} \right) r_1^+ - \left(\frac{p+1}{p} \right) \left(\frac{\bar{p}-1}{\bar{p}} \right) r_1^- \right] \\ &+ \frac{M}{8} (p + \bar{p}) \left[\left(\frac{p-1}{p} \right) \left(\frac{\bar{p}-1}{\bar{p}} \right) r_2^+ - \left(\frac{p+1}{p} \right) \left(\frac{\bar{p}+1}{\bar{p}} \right) r_2^- \right] \quad , \end{aligned}$$

where the following notation have been used

$$\begin{aligned} p &\equiv +\sqrt{1 + 2\mathcal{J}} \\ r_i^\pm &\equiv +\sqrt{\rho^2 + (z - \alpha_i^\pm)^2} \end{aligned} \quad . \quad (33)$$

Oddly enough, for this case the structure of the potential ξ on symmetry axis is defined by the following coefficients m_n

$$m_0 \equiv M \quad , \quad m_1 \equiv iJ \quad , \quad m_k = 0 \quad , \quad \forall k \geq 2 \quad . \quad (34)$$

and so, reproduce only up to order 4 the coefficients that characterize the $M - J$ solution. Multipole moments of this solution turn out to be

$$\begin{aligned}
M_0 &= M & M_6 &= -\frac{1}{33}\mathcal{J}^2 M^7 \\
M_1 &= \mathcal{J} M^2 & M_7 &= \frac{19}{429}\mathcal{J}^3 M^8 \\
M_2 &= 0 & M_8 &= M^9 \left(\frac{1}{143}\mathcal{J}^2 - \frac{53}{3003}\mathcal{J}^4 \right) \\
M_3 &= 0 & M_9 &= M^{10} \left(-\frac{43}{2431}\mathcal{J}^3 + \frac{41}{17017}\mathcal{J}^5 \right) \\
M_4 &= \frac{1}{7}\mathcal{J}^2 M^5 & M_{10} &= M^{11} \left(-\frac{7}{4199}\mathcal{J}^2 + \frac{202}{12597}\mathcal{J}^4 \right) \\
M_5 &= -\frac{3}{21}\mathcal{J}^3 M^6
\end{aligned} \tag{35}$$

It should be noticed that quadrupole moment and octupole moment are zero by contruction. The first multipole moment different from zero is M_4 , which turns out to be proportional to \mathcal{J}^2 , one order higher than the angular momentum in the parameter \mathcal{J} . It is noteworthy that all higher massive moments are proportional to \mathcal{J}^2 and the dynamic moments turn out to be of order 3 in that parameter, and therefore, just the same order than quadrupole moment and octupole moment in the previous case $N = 1$. We will discuss this issue later.

Another interesting property of this solution is its equatorial symmetry which can be inferred from the fact that its odd multipole moments are imaginary quantities (and so, according to notation *FHP* [1], represent dynamic moments) while even multipole moments are real quantities (masive moments). It is an intrinsic characteristic of the procedure used to construct solutions from rational type Ernst's potentials on the symmetry axis. In fact, it can be proved that if coefficients m_n introduced are alternatively real and imaginary quantities, then the resulting Ernst's potential has equatorial symmetry. That occurs because the coefficients a_k and b_k (16) fulfill the next relation $a_k = (-1)^k b_k^*$. Hence, in order to have equatorial symmetry an axisymmetric stationary and asymptotically flat vacuum solution must meet the following necessary sufficient condition [12], [13]:

$$e_+(z) e_+^*(-z) = 1 \quad , \tag{36}$$

where $e_+(z)$ denotes the Ernst's potential on the positive region of the symmetry axis and symbol $*$ denotes complex conjugation.

C) ORDER $N = 3$

In this case we introduce six coefficients m_k according to the ones possessing the $M - J$ solution (3), which means,

$$\begin{aligned} m_0 &\equiv M \quad , \quad m_1 \equiv iJ \quad , \quad m_2 = m_3 = 0 \quad , \\ m_4 &= \frac{1}{7}MJ^2 \quad , \quad m_5 = -i\frac{1}{21}J^3 \quad . \end{aligned} \quad (37)$$

This choice ensures that the multipole moments lesser than M_6 are zero, except for mass and angular momentum.

Then, the Ernst's potential on the symmetry axis can be written as follows

$$e^{(3)}(z) = \frac{P^{(3)}(z)}{Q^{(3)}(z)} \quad , \quad (38)$$

being

$$\begin{aligned} P^{(3)}(z) &= z^3 + Mz^2(-1 + \mathcal{J}/3) + zM^2(-1/7 - 4\mathcal{J}/3) + \\ &\quad + M^3(1/7 + \mathcal{J}/7 - \mathcal{J}^2/3) \\ Q^{(3)}(z) &= z^3 + Mz^2(1 + \mathcal{J}/3) + zM^2(-1/7 + 4\mathcal{J}/3) + \\ &\quad + M^3(-1/7 - \mathcal{J}/7 - \mathcal{J}^2/3) \quad . \end{aligned} \quad (39)$$

The roots of the function $h(z, z)$ (14) corresponding to this potential are

$$\begin{aligned} \alpha_1^\pm &= \pm M \sqrt{A^- + A^+ + \frac{3}{7} + \frac{\mathcal{J}^2}{27}} \\ \alpha_2^\pm &= \pm M \sqrt{-\frac{1}{2}(A^- + A^+) + \frac{3}{7} + \frac{\mathcal{J}^2}{27} + i\frac{1}{2}\sqrt{3}(A^+ - A^-)} \quad , \\ \alpha_3^\pm &= \pm M \sqrt{-\frac{1}{2}(A^- + A^+) + \frac{3}{7} + \frac{\mathcal{J}^2}{27} - i\frac{1}{2}\sqrt{3}(A^+ - A^-)} \end{aligned} \quad (40)$$

where the following notation has been used

$$\begin{aligned} A^\pm &\equiv (a \pm \sqrt{b})^{(1/3)} \\ a &\equiv \frac{8}{343} - \frac{106}{441}\mathcal{J}^2 + \frac{131}{3402}\mathcal{J}^4 + \frac{1}{19683}\mathcal{J}^6 \\ b &\equiv -\frac{256}{50421}\mathcal{J}^2 + \frac{21284}{583443}\mathcal{J}^4 + \frac{71546}{6751269}\mathcal{J}^6 + \frac{5309}{4960116}\mathcal{J}^8 + \frac{1}{177147}\mathcal{J}^{10} \end{aligned} \quad (41)$$

If $|\mathcal{J}| < 1$, which is realistic enough, then $b > 0$ and $a \pm \sqrt{b} > 0$, and so, there are two real roots and two pairs of complex conjugated roots. For the sake of concision we will dispense with the expression resulting for the Ernst's potential.

The coefficients m_k of this exact solution can be calculated by using expressions in [11] to obtain the following ten coefficients

$$\begin{aligned}
m_0 &= M \quad , \quad m_1 = \mathcal{J}M^2 \quad , \quad m_2 = m_3 = 0 \quad , \\
m_4 &= -\frac{1}{7}M^5\mathcal{J}^2 \quad , \quad m_5 = \frac{1}{21}M^6\mathcal{J}^3 \\
m_6 &= -M^7\left(\frac{1}{49}\mathcal{J}^2 + \frac{1}{63}\mathcal{J}^4\right) \\
m_7 &= M^8\left(\frac{5}{147}\mathcal{J}^3 + \frac{1}{189}\mathcal{J}^5\right) \\
m_8 &= M^9\left(-\frac{1}{343}\mathcal{J}^2 - \frac{1}{49}\mathcal{J}^4 - \frac{1}{567}\mathcal{J}^6\right) \\
m_9 &= M^{10}\left(\frac{3}{343}\mathcal{J}^3 + \frac{13}{1323}\mathcal{J}^5 + \frac{1}{1701}\mathcal{J}^7\right) \\
m_{10} &= -M^{11}\left(\frac{1}{2401}\mathcal{J}^2 + \frac{11}{1029}\mathcal{J}^4 - \frac{17}{3969}\mathcal{J}^6 + \frac{1}{5103}\mathcal{J}^8\right)
\end{aligned} \tag{42}$$

According to these expressions the multipole moments are

$$\begin{aligned}
M_0 &= M \quad , \quad M_1 = M^2\mathcal{J} \quad , \quad M_2 = 0 \quad , \quad M_3 = 0 \quad , \quad M_4 = 0 \\
M_5 &= 0 \quad , \quad M_6 = -M^7\left(-\frac{4}{147}\mathcal{J}^2 + \frac{1}{63}\mathcal{J}^4\right) \quad , \quad M_7 = M^8\left(-\frac{12}{539}\mathcal{J}^3 + \frac{1}{189}\mathcal{J}^5\right) \\
M_8 &= M^9\left(-\frac{32}{3773}\mathcal{J}^2 + \frac{554}{63063}\mathcal{J}^4 - \frac{1}{567}\mathcal{J}^6\right) \\
M_9 &= -M^{10}\left(-\frac{26912}{2501499}\mathcal{J}^3 + \frac{9158}{3216213}\mathcal{J}^5 - \frac{1}{1701}\mathcal{J}^7\right) \\
M_{10} &= -M^{11}\left(-\frac{13392}{10081799}\mathcal{J}^2 + \frac{55500}{15842827}\mathcal{J}^4 - \frac{130}{1281987}\mathcal{J}^6 + \frac{1}{5103}\mathcal{J}^8\right)
\end{aligned} \tag{43}$$

We can see that, by construction, this solution obviously possesses a higher number of null multipole moments than the previous solution. Besides, the first multipole moment different from zero, i.e., M_6 , turns out to be proportional to \mathcal{J}^2 again, that is to say, one order less than the angular momentum. Nevertheless, it must be pointed out that its magnitude is not necessarily smaller than the first moment different from zero (M_4) in the previous case. In fact, first multipole moment different from zero for each case is always proportional to \mathcal{J}^2 , and so, we can achieve striking solutions in this process which, in comparison with previous solutions in the sequence, possess moments of a higher multipole order and higher magnitude at the same time.

In order to illustrate the behaviour of the multipole moments in this sequence of exact solutions, we write out the moment of a certain multipole order for each

solution. By way of example, we will compare the moment M_6 of each solution with just the same moment for the case $N = 1$, i.e., Kerr's metric

$$\begin{aligned}
M_6^{(1)} &= M^7 \mathcal{J}^6 = M_6^{Kerr} \\
M_6^{(2)} &= -\frac{1}{33} M^7 \mathcal{J}^2 = -\frac{1}{33} M_6^{Kerr} \frac{1}{\mathcal{J}^4} \\
M_6^{(3)} &= -\frac{1}{49} M^7 \mathcal{J}^2 - \frac{1}{63} M^7 \mathcal{J}^4 = -\frac{1}{49} M_6^{Kerr} \frac{1}{\mathcal{J}^4} - \frac{1}{63} M_6^{Kerr} \frac{1}{\mathcal{J}^2}
\end{aligned} \tag{44}$$

That is to say, the higher approximation degree in the series of solutions, the higher magnitude of the multipole moment. According to (44), the sequence of solutions should have a good behaviour, i.e., progressive diminution of the magnitude of any moment, if the parameter \mathcal{J} were larger than 1 (absolute value), i.e., $J > M^2$, which is not an expected condition for any realistic object.

For these reasons, and others which will be discussed in next section we will introduce a different approach to the Monopole-Dynamic Dipole stationary solution.

4. STATIONARY APROXIMATE M-J SOLUTION

According to the previous section, the construction of an exact stationary and axisimetric solution by Sibgatullin's method requires the structure of Ernst's potential on the symmetry axis as a polynomial ratio. Nevertheless, we have shown that the structure of the Ernst's potential on the symmetry axis corresponding to a solution of type $M - J$ is, in some way, a ratio of series, which means that it cannot be expressed as a polynomials ratio. Hence, although the potential of such a solution were obtained on the symmetry axis, we cannot apply Sibgatullin's method. Besides, the sequence of exact solutions previously proposed approaches the $M - J$ solution in a rather unexpected way, since the magnitude of multipole moments does not decrease while the approximation degree raises.

Therefore, in this section we will proceed to approach the $M - J$ solution in a different manner. We give up looking for exact solutions and propose instead a sequence of approximate solutions as partial sums of power series on the parameter \mathcal{J} .

The expressions in (3) of the coefficients m_n corresponding to the solution $M - J$ lead to an Ernst's potential ξ on the symmetry axis as a power series in the parameter \mathcal{J} . According to this result we will look for solutions in that way. Let us consider the Ernst's equation for the potential ξ and let us assume a solution

of the form

$$\xi \equiv \xi_0 + \sum_{\alpha=1}^{\infty} \xi_{\alpha} \mathcal{J}^{\alpha} \quad , \quad (45)$$

where ξ_0 represents the the Ernst's potential corresponding to Schwarzschild's solution. Imposing this series to verify Ernst equation at each order leads to the following equations concerning the functions ξ_{α}

$$\begin{aligned} (\xi_0^2 - 1) \Delta \xi_{2\alpha+1} - 4\xi_0 \nabla \xi_0 \nabla \xi_{2\alpha+1} + 2\xi_{2\alpha+1} (\nabla \xi_0)^2 &= H_{2\alpha+1} \\ (\xi_0^2 - 1) \Delta \xi_{2\alpha} - 4\xi_0 \nabla \xi_0 \nabla \xi_{2\alpha} + 2\xi_{2\alpha} (\nabla \xi_0)^2 \frac{\xi_0^2 + 1}{\xi_0^2 - 1} &= H_{2\alpha} \end{aligned} \quad , \quad (46)$$

where first equation refers to odd orders and second one to even orders ($\alpha = 1, 2, \dots$); the second members of those equations are given by

$$H_{\alpha} = \sum_{\substack{i+j+k=\alpha \\ i,j,k < \alpha}} (-1)^i [2\xi_i \nabla \xi_j \nabla \xi_k - \xi_i \xi_j \Delta \xi_k] \quad , \quad \alpha > 0 \quad . \quad (47)$$

i.e., equation of order α depends on the previous orders.

The previous equations can be simplified by redefining the functions ξ_{α} as follows

$$\zeta_{\alpha} \equiv \frac{\xi_{\alpha}}{\xi_0^2 - 1} \quad . \quad (48)$$

which leads at each order α to the following equations

$$(\xi_0^2 - 1) \Delta \zeta_{\alpha} + \xi_0^2 \nu^2 \zeta_{\alpha} = \frac{H_{\alpha}}{\xi_0^2 - 1} \quad , \quad \begin{cases} \nu = 0 & , \quad \alpha = \text{even} \\ \nu = 2 & , \quad \alpha = \text{odd} \end{cases} \quad (49)$$

It is easy to solve this equation by writing it in prolate coordinates. As usual the general solution can be obtained by adding a particular solution of inhomogeneous equation to the general solution of whole equation. Moreover we impose a regular behaviour on the symmetry axis ($y = \pm 1$) at least like $1/x$ with respect to the variable z in the neighbourhood of infinity. We then obtain:

$$\begin{aligned} \xi &= \frac{1}{x} + \sum_{\alpha=1}^{\infty} \mathcal{J}^{\alpha} \xi_{\alpha}(x, y) \\ \xi_{\alpha}(x, y) &= \frac{x^2 - 1}{x^2} \left[\zeta_{\alpha}^P(x, y) + \sum_{n=0}^{\infty} h_n^{\alpha} Q_n^{(\nu)}(x) P_n(y) \right] \end{aligned} \quad , \quad (50)$$

where $Q_n^{(\nu)}(x)$ are associated Legendre's functions of second kind, the functions $\zeta_\alpha^P(x, y)$ are particular solutions of the inhomogeneous equations corresponding to each order α , and h_n^α are arbitrary constants.

To describe the Monopole-Dynamic dipole solution $(M - J)$ from the general solution in (50), it is necessary to add as a boundary condition the behaviour of potential ξ on the symmetry axis, which has been defined previously by the series in (5). Hence, we force now the function Φ_α appearing in (5) to agree with the corresponding restriction on the symmetry axis of the function ξ_α of general solution (50), which leads to determine the constants h_n^α .

Previously, the functions Φ_α must be adapted to the structure of the general solution in (50), and so, we begin by taking a factor $\frac{M^2 - z^2}{z^2}$ out of the expression of the function Φ_α ,

$$\Phi_{2n} = \frac{M^2 - z^2}{z^2} \left[\frac{1}{\hat{\lambda}^2 - 1} \sum_{k=2n}^{\infty} G(2n, 2k) \hat{\lambda}^{2k+1} \right] . \quad (51)$$

By carrying out an expansion on the parameter $\hat{\lambda}$ results in

$$\Phi_{2n} = -\frac{M^2 - z^2}{z^2} \sum_{i=0}^{\infty} \sum_{k=2n}^{\infty} G(2n, 2k) \hat{\lambda}^{2k+2i+1} , \quad (52)$$

that is to say,

$$\Phi_{2n} = -\frac{M^2 - z^2}{z^2} \sum_{j=2n}^{\infty} \hat{\lambda}^{2j+1} \sum_{k=2n}^j G(2n, 2k) . \quad (53)$$

Below we write these functions Φ_{2n} in terms of the Legendre's functions of second kind, by using Lemma 4 of the Appendix

$$\Phi_{2n} = -\frac{M^2 - z^2}{z^2} \sum_{j=2n}^{\infty} I_j^{(2n)} \sum_{i=j}^{\infty} (4i+1) L_{2i,2j} Q_{2i}(1/\hat{\lambda}) , \quad (54)$$

where the following notation has been used

$$I_j^{(2n)} \equiv \sum_{k=2n}^j G(2n, 2k) . \quad (55)$$

With respect to the odd functions Φ_{2n+1} we proceed in the same way to obtain mentioned factor and write, in this case, these functions in terms of associated

Legendre's functions of second kind $Q_{2l+1}^{(2)}(1/\hat{\lambda})$, which can be obtained making use of Lemma 5 presented in the Appendix, and so,

$$\Phi_{2\alpha+1} = \frac{z^2 - M^2}{z^2} \sum_{l=2\alpha}^{\infty} \frac{4l+3}{(2l+2)(2l+1)} Q_{2l+1}^{(2)}(1/\hat{\lambda}) \sum_{n=2\alpha}^l \frac{2l+2n+1}{(2n+1)!!} L_{2l,2n} I_n^{(2\alpha+1)} \quad (56)$$

with the following notation

$$I_n^{(2\alpha+1)} \equiv \sum_{k=2\alpha}^n G(2\alpha+1, 2k+1) \quad . \quad (57)$$

At this point we proceed to determine the constants h_n^α of the general solution (50) which correspond to the $M - J$ solution . In that way, we choose particular solutions of inhomogeneous equations (49) as follows

$$\begin{aligned} \zeta_{2\alpha}^P &= \sum_{l=0}^{\infty} (4l+1) Q_{2l}(x) S_{2l}(y) \\ \zeta_{2\alpha+1}^P &= \sum_{l=0}^{\infty} (4l+3) Q_{2l+1}^{(2)}(x) S_{2l+1}(y) \end{aligned} \quad , \quad (58)$$

$S_a(y)$ being polynomials in the angular variable.

Comparing general solution (50), evaluated on the symmetry axis, with the previous expressions (54) and (56) gives

$$\begin{aligned} h_{2n+1}^{2\alpha} &= 0 \\ h_{2n}^{2\alpha} &= -(4n+1) S_{2n}(1) \quad , \quad n < 2\alpha \\ h_{2n}^{2\alpha} &= -(4n+1) \left[S_{2n}(1) + \sum_{k=2\alpha}^n L_{2n,2k} I_k^{(2\alpha)} \right] \quad , \quad n \geq 2\alpha \\ h_{2n}^{2\alpha+1} &= 0 \\ h_{2n+1}^{2\alpha+1} &= -(4n+3) S_{2n+1}(1) \quad , \quad n < 2\alpha \\ h_{2n+1}^{2\alpha+1} &= -(4n+3) \left[S_{2n+1}(1) + \frac{1}{(2n+1)(2n+2)} \sum_{l=2\alpha}^n \frac{2n+2l+1}{(2l+1)!!} L_{2n,2l} I_l^{(2\alpha+1)} \right] \\ &\quad n \geq 2\alpha \end{aligned} \quad (59)$$

Let us construct explicitly the first orders of the solution $M - J$. Obviously, the order zero contribution to the solution must be Schwarzschild's solution, since

taking the parameter $\mathcal{J} = 0$ leads us to consider the mass as the unique multipole moment. The Ernst's potential ξ of Schwarzschild's solution ($\xi_0 = 1/x$) equals the structure described on the symmetry axis (5).

A) FIRST ORDER.

The first contribution on the parameter \mathcal{J} should be a solution of the first equation in (46) turns out to be homogeneous at order one, and hence

$$\xi_1 = \frac{1-x^2}{x^2} \sum_{l=0}^{\infty} h_l^1 Q_l^{(2)}(x) P_l(y) \quad . \quad (60)$$

This expression on the symmetry axis, gives

$$\xi_1(y=1) = 2\hat{\lambda}h_0^1 + 2\hat{\lambda}^2h_1^1 + (\hat{\lambda}^2 - 1)^2 \sum_{l=2}^{\infty} h_l^1 Q_l^{(2)}(1/\hat{\lambda}) \quad . \quad (61)$$

In addition, the first contribution to the $M - J$ solution must agree with (5) on the symmetry axis and so, the only solution corresponds to the following choice of constants

$$h_0^1 = 0 \quad , \quad h_1^1 = \frac{1}{2} \quad , \quad h_l^1 = 0, \quad l \geq 2 \quad , \quad (62)$$

which means,

$$\xi_1(x, y) = \frac{1}{2} \frac{1-x^2}{x^2} Q_1^{(2)}(x) P_1(y) = \frac{y}{x^2} \quad . \quad (63)$$

It must be noted that this approximate solution is the same as the one arising from the expansion of Kerr's metric on parameter \mathcal{J} up to the first order.

B) SECOND ORDER.

A particular solution of the inhomogeneous equation (46), corresponding to order two, results in

$$\zeta_2^{inh} = \left(\frac{x}{2} - \frac{y^2}{x}\right) \frac{1}{1-x^2} \quad . \quad (64)$$

and in view of Lemma 4 of the Appendix, this could be rewritten in terms of associated Legendre's functions in the following way

$$\zeta_2^{inh} = \sum_{n=0}^{\infty} Q_{2n}(x)(4n+1) \left[\frac{1}{2} - y^2(1 - L_{2n,0}) \right] \quad . \quad (65)$$

By substituting this particular solution in the expressions (59) we determine the constants of the contribution of the second order to the solution, i.e.,

$$\begin{aligned} h_0^2 &= \frac{1}{2} \quad , \quad h_2^2 = 5 \left(L_{2,0} - \frac{1}{2} \right) \\ h_{2n}^2 &= (4n+1) \left[-\frac{1}{2} + L_{2n,0} - \sum_{k=2}^n L_{2n,2k} I_k^{(2)} \right] \quad , \quad n \geq 2 \end{aligned} \quad (66)$$

Now, taking the expression $G(2, 2j)$ (8) into account we can write term $I_k^{(2)}$ as a sum of irreducible fractions in the following form

$$I_k^{(2)} = \frac{(k+3)(k-1)}{(2k+3)(2k+1)} = \frac{1}{4} - \frac{15}{8} \frac{1}{2k+1} + \frac{15}{8} \frac{1}{2k+3} \quad (67)$$

which implies that $I_0^{(2)} = -1$ and $I_1^{(2)} = 0$. Therefore constants h_{2n}^2 with $n \geq 2$ can be written as follows

$$h_{2n}^2 = (4n+1) \left[-\frac{1}{2} - \sum_{k=0}^n L_{2n,2k} I_k^{(2)} \right] \quad (68)$$

Considering now expression (67) we have

$$h_{2n}^2 = (4n+1) \left[-\frac{1}{2} + \frac{15}{8} \sum_{k=0}^n \frac{L_{2n,2k}}{2k+1} - \frac{15}{8} \sum_{k=0}^n \frac{L_{2n,2k}}{2k+3} - \frac{1}{4} \sum_{k=0}^n L_{2n,2k} \right] \quad (69)$$

and making use of Lemma 2 of the Appendix, and the ortonormality of the Legendre's polynomials, we have finally the following expressions

$$h_0^2 = \frac{1}{2} \quad , \quad h_2^2 = -5 \quad , \quad h_{2n}^2 = -\frac{3}{4}(4n+1), \quad n \geq 2 \quad (70)$$

Hence, the contribution of order 2 to the solution $M - J$ can be written as follows

$$\begin{aligned} \xi_2(x, y) &= \frac{1}{2x} - \frac{y^2}{x^3} + \frac{1-x^2}{x^2} \left[\frac{1}{2} Q_0(x) P_0(y) - 5 Q_2(x) P_2(y) - \right. \\ &\quad \left. - \sum_{n=2}^{\infty} \frac{3}{4} (4n+1) Q_{2n}(x) P_{2n}(y) \right] \quad , \end{aligned} \quad (71)$$

an expression which can be simplified by making use of Heine's identity [14]:

$$\frac{x}{x^2 - y^2} = \sum_{n=0}^{\infty} (4n+1) Q_{2n}(x) P_{2n}(y) \quad , \quad (72)$$

which leads to the following expression

$$\xi_2(x, y) = \frac{1}{2x} - \frac{y^2}{x^3} + \frac{1-x^2}{x^2} \left[\frac{5}{4} Q_0(x) P_0(y) - \frac{5}{4} Q_2(x) P_2(y) - \frac{3}{4} \frac{x}{x^2 - y^2} \right], \quad (73)$$

that is to say

$$\xi_2(x, y) = \frac{5}{4} \frac{x^2 - 1}{x^2} \left[Q_0(x) P_0(y) - Q_2(x) P_2(y) - \frac{x}{x^2 - y^2} \right] + \xi_2^{Kerr}, \quad (74)$$

where ξ_2^{Kerr} is the order 2 in the expansion of Kerr's metric for the parameter \mathcal{J} .

The multipole moments can be calculated by *FHP* algorithm from coefficients m_n . For the solution up to order 1, i.e., $\xi_{M-J}^{(1)} \equiv \xi_0 + \mathcal{J} \xi_1$, all coefficients m_n with $n \geq 2$ are zero (i.e., of higher order than \mathcal{J}). So, its multipole moments equal those in the exact solution presented in the previous section whose potential $e(z)$ was a ratio of polynomials of order 2, that is to say,

$$\begin{aligned} M_0 &= M, \quad M_1 = \mathcal{J} M^2, \quad M_2 = 0, \quad M_3 = 0, \\ M_4 &= \frac{1}{7} \mathcal{J}^2 M^5, \quad M_5 = -\frac{3}{21} \mathcal{J}^3 M^6, \quad M_6 = -\frac{1}{33} \mathcal{J}^2 M^7 \\ M_7 &= \frac{19}{429} \mathcal{J}^3 M^8, \quad M_8 = -M^9 \left(-\frac{1}{143} \mathcal{J}^2 + \frac{53}{3003} \mathcal{J}^4 \right) \\ M_9 &= M^{10} \left(-\frac{43}{2431} \mathcal{J}^3 + \frac{41}{17017} \mathcal{J}^5 \right), \quad M_{10} = M^{11} \left(-\frac{7}{4199} \mathcal{J}^2 + \frac{202}{12597} \mathcal{J}^4 \right). \end{aligned} \quad (75)$$

With respect to the $M - J$ solution up to the second order, i.e., $\xi_{M-J}^{(2)} \equiv \xi_0 + \mathcal{J} \xi_1 + \mathcal{J}^2 \xi_2$, we have the following multipole moments

$$\begin{aligned} M_0 &= M, \quad M_1 = \mathcal{J} M^2, \quad M_2 = 0, \quad M_3 = 0, \quad M_4 = 0 \\ M_5 &= -\frac{1}{21} \mathcal{J}^3 M^6, \quad M_6 = 0, \quad M_7 = -\frac{59}{3003} \mathcal{J}^3 M^8, \quad M_8 = -\frac{2}{231} \mathcal{J}^4 M^9 \\ M_9 &= M^{10} \left(\frac{41}{17017} \mathcal{J}^5 + \frac{593}{51051} \mathcal{J}^3 \right), \quad M_{10} = -\frac{49873}{6789783} \mathcal{J}^4 M^{11} \end{aligned} \quad (76)$$

It can be seen from (75) and (76) that the higher the order of approximation to $M - J$ solution, the higher (one order more) the order in the parameter \mathcal{J} of its non vanishing multipole moments.

The structure of coefficients m_k in terms of the parameter \mathcal{J} shows which is the order n of such coefficients that possess a contribution of order α in \mathcal{J} .

In fact, the first contribution to an even power in the parameter, arises from the coefficient of order 2α . If α is odd, such contribution arise from the coefficient of order $2\alpha - 1$. Since the relation between the multipole moment and the coefficient m_k of same order is linear we can conclude that the multipole moments of the solution which approche the $M - J$ solution up to the order α in the parameter \mathcal{J} have the following characteristics:

1) If the order α is even, all its massive multipole moments up to $M_{2\alpha+2}$ (incl.) will be zero, the following ones being at least of order $\alpha + 2$ in parameter \mathcal{J} . With regard to its dynamic multipole moments they will be zero up to $M_{2\alpha-1}$ (incl.), and the following ones will be at least of order $\alpha + 1$ in parameter \mathcal{J} .

2) If the order of approximation α is odd, then all massive multipole moments will be zero up to $M_{2\alpha}$ (incl.) and the ones higher than that will be at least of order $\alpha + 1$, whereas the dynamic moments will be zero up to $M_{2\alpha+1}$ and the following moments will be at least of order $\alpha + 2$.

These results show, in the same way as the static $M - Q$ solution [5], that it is possible to understand the series in (45) in terms of the perturbations theory. Each partial sum of that series is a better approximation to the solution which only has mass and angular momentum, since the multipole moments higher than this are either zero or have an order in the parameter \mathcal{J} higher than the one of the approximation. In addition, unlike the solutions in the previous section, the multipole moment of a certain order is progressively smaller as the order of approximation in the series (45) grows.

APPENDIX

In this Appendix we will enunciate a sequence of Lemmas about several properties of the Legendre's polynomials and the associated Legendre's functions of second kind. Results of these Lemmas are probably well known, but proofs for them can be easily obtained considering some results of various Lemmas appearing in [5].

Let us P_{2n} be a Legendre's polynomial of even order in an arbitrary variable

$$P_{2n}(\zeta) = \sum_{k=0}^n L_{2n,2k} \zeta^{2k} \quad , \quad (A.1)$$

where its coefficients have the following expression

$$L_{2n,2k} \equiv (-1)^{n-k} 2^{k-n} \frac{(2n+2k-1)!!}{(n-k)!(2k)!} \quad . \quad (A.2)$$

Let us consider the development of an arbitrary variable to an even power in terms of Legendre's polynomials in that variable, i.e.,

$$\zeta^{2n} = \sum_{k=0}^{\infty} C_{2n,2k} P_{2k}(\zeta) \quad , \quad (A.3)$$

where coefficients $C_{2n,2k}$ can be obtained by integration from the following expression,

$$C_{2n,2k} = \frac{4k+1}{2} \int_{-1}^1 P_{2k}(\zeta) \zeta^{2n} d\zeta \quad , \quad (A.4)$$

and so, for several values of indexes k and n such a coefficients turns out to be

$$C_{2n,2k} = \begin{cases} (4k+1) \frac{2n!}{(2n-2k)!!(2n+2k+1)!!} & : k \leq n \\ 0 & : k > n \end{cases} \quad . \quad (A.5)$$

Lemma 1. The following orthogonality relation is satisfied:

$$\sum_{j=0}^k L_{2k,2j} C_{2j,2n} = \delta_{kn} \quad . \quad (A.6)$$

Corolary: It can be deduced evidently another orthogonality relation

$$\sum_{k=n}^j L_{2k,2n} C_{2j,2k} = \delta_{jn} \quad (A.7)$$

Lemma 2. For all pair of positive entire numbers n and k such that $n < k$, the following equality is verified

$$\sum_{j=0}^k \frac{L_{2k,2j}}{2n+2j+1} = 0 \quad . \quad (A.8)$$

Lemma 3. For all pair of positive entire numbers α and j the following equality is verified:

$$\sum_{n=2\alpha}^{\infty} \frac{\hat{\lambda}^{2n+1}}{2n+2j+1} = \hat{\lambda}^{4\alpha} \sum_{n=0}^{j+2\alpha} C_{2j+4\alpha,2n} Q_{2n}(1/\hat{\lambda}) \quad , \quad \hat{\lambda} \equiv \frac{M}{z} \quad . \quad (A.9)$$

Lemma 4. For all positive entire number n following is verified

$$\frac{1}{x^{2n+1}} = \sum_{k=n}^{\infty} (4k+1) L_{2k,2n} Q_{2k}(x) \quad (A.10a)$$

$$\frac{1}{x^{2n}} = \sum_{k=n-1}^{\infty} (4k+3) L_{2k+1,2n-1} Q_{2k+1}(x) \quad . \quad (A.10b)$$

Lemma 5. For all positive entire number n following is verified

$$\frac{1}{x^{2n}} = \frac{1}{(2n-1)!!} \sum_{k=n-1}^{\infty} (4k+3) Q_{2k+1}^{(2)}(x) L_{2k,2n-2} \frac{2k+2n-1}{(2k+2)(2k+1)} \quad . \quad (A.11)$$

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